BAREKENG: Journal of Mathematics and Its Application
September 2022 Volume 16 Issue 3 Page 829–834
P-ISSN: 1978-7227 E-ISSN: 2615-3017
https://doi.org/10.30598/barekengvol16iss3pp829-834

AN EXISTENCE AND UNIQUENESS OF THE WEAK SOLUTION OF THE DIRICHLET PROBLEM WITH THE DATA IN MORREY SPACES

Nicky K. Tumalun *

Mathematics Department, Faculty of Mathematics and Natural Sciences, Universitas Negeri Manado
Tataran-Remboken St., Kec. Tondano Selatan, Kabupaten Minahasa, 95681, Indonesia

*Corresponding author's e-mail: * nickytumalun@unima.ac.id*

Abstract. *Let $n-2<\lambda<n$, f be a function in Morrey spaces $L^{1,\lambda}(\Omega)$, and the equation*

$$\begin{cases} Lu = f, \\ u \in W_0^{1,2}(\Omega), \end{cases}$$

be a Dirichlet problem, where Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 3$, and L is a divergent elliptic operator. In this paper, we prove the existence and uniqueness of this Dirichlet problem by directly using the Lax-Milgram Lemma and the weighted estimation in Morrey spaces.

Keywords: *Morrey spaces, Dirichlet problem, elliptic equations.*

Article info:

Submitted: 4th April 2022 *Accepted: 9th July 2022*

How to cite this article:
N. K. Tumalun, "AN EXISTENCE AND UNIQUENESS OF THE WEAK SOLUTION OF THE DIRICHLET PROBLEM WITH THE DATA IN MORREY SPACES", *BAREKENG: J. Math. & App.*, vol. 16, iss. 3, pp. 829-834, September, 2022.

1. INTRODUCTION

Let $\boldsymbol{\Omega}$ be a bounded and open subset of $\mathbb{R}^{\boldsymbol{n}}$, where $\boldsymbol{n} \geq \boldsymbol{3}$, and $\boldsymbol{l}$ be the diameter of $\boldsymbol{\Omega}$. For every $\boldsymbol{a} \in \boldsymbol{\Omega}$ and $\boldsymbol{r} > \boldsymbol{0}$, we define

$$B(a,r) = \{y \in \mathbb{R}^n : |y-a| < r\},$$

and

$$\Omega(a,r) = \Omega \cap B(a,r) = \{y \in \Omega : |y-a| < r\}.$$

The Morrey spaces $L^{p,\lambda}(\Omega)$ is defined to be the set of all functions $f \in L^p(\Omega)$ which satisfy

$$\|f\|_{L^{p,\lambda}} = \sup_{a\in\Omega, r>0} \left(\frac{1}{r^\lambda} \int_{\Omega(a,r)} |f(y)|^p dy \right)^{\frac{1}{p}} < \infty,$$

for $1 \leq p < \infty$ and $0 \leq \lambda \leq n$. This Morrey spaces were introduced by C. B. Morrey [1] and still attracted the attention of many researcher to investigate its inclusion properties or application in partial differential equation [2, 3, 4, 5, 6, 7, 8].

Let $W^{1,2}(\Omega)$ be the Sobolev space equipped by the norm

$$\|u\|_{W^{1,2}(\Omega)} = \left(\|u\|^2{}_{L^2(\Omega)} + \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|^2{}_{L^2(\Omega)} \right)^{\frac{1}{2}} = \left(\int_\Omega |u|^2 + \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^2 \right)^{\frac{1}{2}}$$

for every $u \in W^{1,2}(\Omega)$. The closure of $C_0^\infty(\Omega)$ under the Sobolev norm is denoted by $W_0^{1,2}(\Omega)$. It is well known that $W_0^{1,2}(\Omega)$ is a Hilbert space, that is, the Sobolev norm is generated by an inner or scalar product on $W_0^{1,2}(\Omega)$.

We consider the following second order divergent elliptic operator

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{i,j} \frac{\partial u}{\partial x_i} \right), \tag{1}$$

where $u \in W_0^{1,2}(\Omega)$,

$$a_{i,j} \in L^\infty(\Omega), \qquad i,j = 1, \dots, n, \tag{2}$$

and there exists $\nu > 0$ such that

$$\nu|\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j, \tag{3}$$

for every $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ and for almost every $x \in \Omega$.

Let $f \in L^{1,\lambda}(\Omega)$. In this paper, we will investigate the existence and uniqueness of the weak solution to the equation

$$\begin{cases} Lu = f, \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{4}$$

where L is defined by (1) and the λ satisfies a certain condition. The Eq. (4) is called the Dirichlet problem.

The function $u \in W_0^{1,2}(\Omega)$ is called the weak solution of the Dirichlet problem (4) if

$$\int_\Omega \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial \phi(x)}{\partial x_j} dx = \int_\Omega f(x)\phi(x)dx, \tag{5}$$

for every $\phi \in W_0^{1,2}(\Omega)$.

Recently, Tumalun and Tuerah [9], continue the work of Di Fazio [10, 11], proved that the weak solution of gradient of (4) belongs to some weak Morrey spaces by assuming $f \in L^{1,\lambda}(\Omega)$ for $0 < \lambda < n-2$. Notice that, the result in [9], generalized by themselves in [12]. In [9, 10, 11], the authors used a representation of the weak solution, which involves the Green function [13], and proved that this representation satisfies (5) to show the existence of the weak solution of (4). Cirmi et. al [14] proved that the weak solution of (4) exists and unique, and its gradient belongs to some Morrey spaces, where they assumed $f \in L^{1,\lambda}(\Omega)$ for $n-2 < \lambda < n$. The proof of the existence and uniqueness of the weak solution, which is done by Cirmi et. al, used an approximation method.

By assuming $f \in L^{1,\lambda}(\Omega)$, for $n-2<\lambda<n$, in this paper we will give a direct proof of the existence and uniqueness of the weak solution of the Dirichlet problem (4). Our method uses a functional analysis tool, i.e. the Lax-Millgram lemma, combining with a weighted embeddings in Morrey and Sobolev spaces.

2. RESEARCH METHODS

The constant $\boldsymbol{C}=\boldsymbol{C}(\boldsymbol{\alpha},\boldsymbol{\beta},\ldots,\boldsymbol{\gamma})$, which appears throughout this paper, denotes that it is dependent on $\boldsymbol{\alpha},\boldsymbol{\beta},\ldots$, and $\boldsymbol{\gamma}$. The value of this constant may vary from line to line whenever it appears in the theorems or proofs.

Our method relies on functional analysis tools, that is Lax-Milgram lemma, that we will state in this section. We start by write down some properties related to Lax-Milgram lemma.

Let H be a Hilbert space with norm $\|\cdot\|$ and $\boldsymbol{B}: H\times H\to\mathbb{R}$ be a bilinear mapping. The map $\boldsymbol{B}$ is called continuous if there exists a constant $C_1>0$ such that

$$|\boldsymbol{B}(u,w)|\le C_1\|u\|\|w\|,$$

for all $u,w\in H$, and called coercive if there exists a constant $C_2>0$ such that

$$\boldsymbol{B}(u,u)\ge C_2\|u\|^2,$$

for all $u\in H$.

The following lemma is known as Lax-Milgram lemma and H is the Hilbert space with norm $\|\cdot\|$. We refer to [15] for its proof.

Lemma 1 (Lax-Milgram). *Let $\boldsymbol{B}: H\times H\to\mathbb{R}$ be a continuous and coercive bilinear mapping. Then, for every bounded linear functional $F: H\to\mathbb{R}$, there exists a unique element $u\in H$ such that*

$$F(w)=\boldsymbol{B}(u,w),$$

for every $w\in H$.

We associate the operator L with the mapping $\boldsymbol{B}: W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega)\to\mathbb{R}$ defined by the formula

$$\boldsymbol{B}(u,\phi)=\int_\Omega\sum_{i,j=1}^n a_{i,j}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial\phi(x)}{\partial x_j}dx. \tag{6}$$

For $n-2<\lambda<n$ and $f\in L^{1,\lambda}(\Omega)$, we define $F_f: W_0^{1,2}(\Omega)\to\mathbb{R}$ by the formula

$$F_f(\phi)=\int_\Omega f(x)\phi(x)dx. \tag{7}$$

By the linearity of the weak derivative and integration, it is easy to show that $\boldsymbol{B}$, defined by (6), is a bilinear mapping. Notice that, according to (5), (6), and (7), $u\in W_0^{1,2}(\Omega)$ is a weak solution of (4) if

$$\boldsymbol{B}(u,\phi)=F_f(\phi), \tag{8}$$

for every $\phi\in W_0^{1,2}(\Omega)$.

Now we state the following two theorems regarding to the estimation for any functions in $W_0^{1,2}(\Omega)$, that we will need later. The first theorems called Poincaré's inequality (see [15] for its proof) and the second theorem called sub representation formula (see [16] for its proof).

Theorem 1 (Poincaré's Inequality). *If $u\in W_0^{1,2}(\Omega)$, then there exists a positive constant $C=C(l)$ such that*

$$\int_\Omega|u|^2\le C\int_\Omega|\nabla u|^2.$$

Theorem 2 (Sub Representation Formula). *If $u\in W_0^{1,2}(\Omega)$, then there exists a positive constant $C=C(n)$ such that*

$$|u(x)|\le C\int_\Omega\frac{|\nabla u(y)|}{|x-y|^{n-1}}dy,$$

for a. e. $x\in\Omega$.

We close this section by state the following Theorem which slightly modified from [17].

Theorem 3. *Let $n-2<\lambda<n$. If $f \in L^{1,\lambda}(\Omega)$, then there exists a positive constant $C=C(n,\lambda,l)$ such that*

$$\int_{\Omega} \frac{|f(x)|}{|z-x|^{n-2}} dx \leq C\|f\|_{L^{1,\lambda}},$$

for every $z \in \Omega$.

3. RESULTS AND DISCUSSION

To start our discussion, we prove that the bilinear mapping $\boldsymbol{B}$ defined by (6) is continuous and coercive.

Lemma 2. *The mapping $\boldsymbol{B}$ defined by (6) is continuous and coercive*.

Proof. Let $u \in W_0^{1,2}(\Omega)$. We first prove the coercivity property. By using (3) and then Poincaré's inequality, we have

$$B(u,u)=\int_{\Omega} \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial u(x)}{\partial x_j} dx \geq \nu \int_{\Omega} |\nabla u|^2$$

$$\geq \frac{\nu}{2} \int_{\Omega} |\nabla u|^2 + C \int_{\Omega} |u|^2 \geq C\left(\|u\|_{W^{1,2}(\Omega)}\right)^2,$$

where $C=C(\nu,l)$ is a positive constant.

Now, we prove the continuity property. Let $u,\phi \in W_0^{1,2}(\Omega)$. Note that,

$$A=\sum_{i,j=1}^{n} \|a_{i,j}\|_{L^{\infty}(\Omega)}$$

according to (2). By using Hölder's inequality, we have

$$|B(u,\phi)| \leq \int_{\Omega} \sum_{i,j=1}^{n} \|a_{i,j}\|_{L^{\infty}(\Omega)} \left|\frac{\partial u(x)}{\partial x_i}\right| \left|\frac{\partial \phi(x)}{\partial x_j}\right| dx$$

$$\leq A \int_{\Omega} |\nabla u(x)||\nabla \phi(x)| dx$$

$$\leq A\left(\int_{\Omega} |\nabla u(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla u(x)|^2 dx\right)^{\frac{1}{2}}$$

$$\leq A\|u\|_{W^{1,2}(\Omega)} \|\phi\|_{W^{1,2}(\Omega)}.$$

This completes the proof. □

We need the theorem below to prove that the function F_f defined by (7) is a bounded linear functional. This theorem states about a weighted estimation in Morrey spaces where the weight in Sobolev spaces. The proof of this theorem was given in [11]. However, the given proof did not complete. Here we give the complete proof.

Theorem 4. *Let $n-2<\lambda<n$. If $f \in L^{1,\lambda}(\Omega)$, then there exists a positive constant $C=C(n,\lambda,l)$ such that,*

$$\int_{\Omega} |fu| \leq C\|f\|_{L^{1,\lambda}(\Omega)} \|u\|_{W^{1,2}(\Omega)}$$

for every $u \in W_0^{1,2}(\Omega)$.

Proof. Let $u \in W_0^{1,2}(\Omega)$. According to the sub representation formula of u and Hölder's inequality, we have

$$\begin{aligned}\int_\Omega |f(x)u(x)|dx &\le C(n)\int_\Omega |f(x)|\left(\int_\Omega \frac{|\nabla u(y)|}{|x-y|^{n-1}}dy\right)dx\\ &= C(n)\int_\Omega |\nabla u(y)|\left(\int_\Omega \frac{|f(x)|}{|x-y|^{n-1}}dx\right)dy\\ &\le C(n)\|\nabla u\|_{L^2(\Omega)}\left(\int_\Omega \left(\int_\Omega \frac{|f(x)|}{|x-y|^{n-1}}dx\right)^2 dy\right)^{\frac{1}{2}}. \qquad (9)\end{aligned}$$

Notice that

$$\int_\Omega |f(z)|dz \le C(n,\lambda,l)\|f\|_{L^{1,\lambda}(\Omega)}. \qquad (10)$$

Hence

$$\begin{aligned}\int_\Omega \left(\int_\Omega \frac{|f(x)|}{|x-y|^{n-1}}dx\right)^2 dy &= \int_\Omega \left(\int_\Omega \frac{|f(z)|}{|z-y|^{n-1}}dz\right)\left(\int_\Omega \frac{|f(x)|}{|x-y|^{n-1}}dx\right)dy\\ &= \int_\Omega\int_\Omega |f(z)||f(x)|\left(\int_\Omega \frac{1}{|z-y|^{n-1}|x-y|^{n-1}}dy\right)dxdz\\ &\le C(n)\int_\Omega\int_\Omega |f(z)||f(x)|\frac{1}{|z-x|^{n-2}}dxdz\\ &= C(n)\int_\Omega |f(z)|\left(\int_\Omega \frac{|f(x)|}{|z-x|^{n-2}}dx\right)dz\\ &\le C(n,\lambda,l)\|f\|_{L^{1,\lambda}(\Omega)}\int_\Omega |f(z)|\,dz\\ &\le C(n,\lambda,l)\left(\|f\|_{L^{1,\lambda}(\Omega)}\right)^2 \qquad (11)\end{aligned}$$

by virtue of Theorem 3 and (10). Combining (9) and (11), we obtain

$$\begin{aligned}\int_\Omega |f(x)u(x)|dx &\le C(n)\|\nabla u\|_{L^2(\Omega)}\left(\int_\Omega \left(\int_\Omega \frac{|f(x)|}{|x-y|^{n-1}}dx\right)^2 dy\right)^{\frac{1}{2}}\\ &\le C(n,\lambda,l)\|\nabla u\|_{L^2(\Omega)}\|f\|_{L^{1,\lambda}(\Omega)}\\ &\le C(n,\lambda,l)\|u\|_{W^{1,2}(\Omega)}\|f\|_{L^{1,\lambda}(\Omega)}.\end{aligned}$$

The theorem is proved. □

From Theorem 4, we obtain the following corollary.

Corollary 1. *Let $n-2<\lambda<n$. The mapping $F_f: W_0^{1,2}(\Omega)\to\mathbb{R}$ defined by (7) is a bounded linear functional.*

Proof. According to the linearity of integration, F_f is a linear functional on $W_0^{1,2}(\Omega)$. For every $u\in W_0^{1,2}(\Omega)$, Theorem 4 gives us

$$\left|F_f(u)\right| \le \int_\Omega |fu| \le C\|u\|_{W^{1,2}(\Omega)},$$

where the positive constant $C = C\left(n,\lambda,l,\|f\|_{L^{1,\lambda}(\Omega)}\right)$. This means F_f is also bounded and the proof is complete. □

Combining Lemma 2, Corollary 1, and Lax-Milgram lemma, we now state the following existence and uniqueness of the weak solution of the Dirichlet problem (4).

Theorem 5. *Let $n-2<\lambda<n$ and $f\in L^{1,\lambda}(\Omega)$ in Dirichlet problem (4). Then there exists a unique element $u\in W_0^{1,2}(\Omega)$ which is the weak solution of the Dirichlet problem (4).*

4. CONCLUSIONS

The Dirichlet problem (4) has a unique weak solution by assuming the data belongs some Morrey spaces. This fact can be proved by using functional analysis tool, i.e. the Lax-Milgram lemma, combining with the weighted embedding in Morrey spaces where the weight in Soblev spaces.

AKNOWLEDGEMENT

The author thanks LPPM Unima for supporting this research in 2022.

REFERENCES

[1] C. B. Morrey, "On the solutions of quasi-linear elliptic partial differential equations," *Trans. Amer. Math. Soc.*, vol. 43, pp. 126-166, 1938.

[2] P. E. A. Tuerah and N. K. Tumalun, "Some notes on the inclusion between Morrey spaces," *J. Math. Inequal.*, vol. 16, no. 1, pp. 355-362, 2022.

[3] N. K. Tumalun, "Proper inclusion between vanishing Morrey spaces and Morrey spaces," *Tensor: Pure and Applied Mathematics Journal*, vol. 2, no. 1, pp. 1-4, 2021.

[4] R. Kawasumi and E. Nakai, "Pointwise multipliers on weak Morrey spaces," *Anal. Geom. Metr. Spaces*, vol. 8, no. 1, pp. 363-381, 2020.

[5] N. K. Tumalun, D. I. Hakim and H. Gunawan, "Inclusion between generalized Stummel classes and other function spaces," *Math. Inequal. Appl.*, vol. 23, no. 2, pp. 547-562, 2020.

[6] N. K. Tumalun and H. Gunawan, "Morrey spaces are embedded between weak Morrey spaces and Stummel classes," *J. Indones. Math. Soc.*, vol. 25, no. 3, pp. 203-209, 2019.

[7] N. K. Tumalun, H. Gunawan and J. Lindiarni, "On the inclusion between weak Lebesgue spaces and Stummel classes," in *5th ICRIEMS*, Yogyakarta, 2018.

[8] N. K. Tumalun, D. I. Hakim and H. Gunawan, "Some function spaces and their applications to elliptic partial differential equations," *Math. Vesnik (in press)*, 2022.

[9] N. K. Tumalun and P. E. A. Tuerah, "A regularity of the weak solution gradient of the Dirichlet problem for divergent form elliptic equations in Morrey spaces," *Aust. J. Math. Anal. Appl.*, vol. 18, no. 2, pp. 14.1-14.7, 2021.

[10] G. Di Fazio, "Regularity for elliptic equations under minimal assumptions," *J. Indones. Math. Soc.*, vol. 26, no. 1, pp. 101-127, 2020.

[11] G. Di Fazio, "On Dirichlet problem in Morrey spaces," *J. Differential and Integral Equations*, vol. 6, pp. 383-391, 1993.

[12] N. K. Tumalun and P. E. A. Tuerah, "A regularity of the Dirichlet problem with the data belongs to generalized Morrey spaces," in *AIP Proceeding (to appear)*, Semarang, 2021.

[13] M. Gruter and K. O. Widman, "The Green function for uniformly elliptic equations," *Manuscripta Math.*, vol. 37, pp. 303-342, 1982.

[14] G. R. Cirmi, S. D'Asero and S. Leonardi, "Morrey estimates for a class of elliptic equations with drift terms," *Adv. Nonlinear Anal.*, vol. 9, pp. 1333-1350, 2020.

[15] H. Brezis, Functional Analysis, Sobolev Spaces, and Partial Differential Equations, New York: Springer, 2011.

[16] R. Wheeden and A. Zygmund, Measure and Integral. An Introduction to Real Analysis. Second Edition, Boca Raton: Taylor & Francis Group, 2015.

[17] G. Di Fazio, "Holder continuity of the solutions for some Schrodinger equations," *Rend. Sem. Math. Padova*, vol. 79, pp. 173-183, 1988.